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$p(x)$-Laplacian equations satisfying Cerami condition ☆

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Abstract

The goal of this paper is to prove the existence and multiplicity of solutions for the $p(x)$-Laplacian equations without the well-known Ambrosetti–Rabinowitz type growth conditions. The critical point theorems with Cerami condition are used.

Keywords: $p(x)$-Laplacian; Variable exponent Sobolev space; Critical point theorem; Cerami condition

1. Introduction

The differential equations and variational problems with nonstandard growth conditions have been studied in recent years. Some results on these problems have been obtained. For example, we refer to [6–13,19] and references therein.

At first, we study the existence and multiplicity of solutions for the $p(x)$-Laplacian Dirichlet problems of the following form:

$$\begin{cases} -\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p(x) > 1$, $\forall x \in \Omega$, and $p \in C(\overline{\Omega})$.

In this paper, we establish the following basic assumptions and notations:

Let $F(x,s) = \int_0^s f(x,t)\,dt$, $|E|$ be the Lebesgue measure of E, and the family of functions $\mathcal{F} = \{G_\lambda \mid G_\lambda(x,t) = f(x,t)t - \lambda F(x,t), \lambda \in [p^-, p^+]\}$, where $p^- = \inf_{x\in\Omega} p(x)$, $p^+ = \sup_{x\in\Omega} p(x)$. Noticing that when $p(x) \equiv p$ is a constant, $\mathcal{F} = \{f(x,t)t - pF(x,t)\}$ contains only one element.

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and

$$\bigl|f(x,t)\bigr| \leqslant C_1 + C_2|t|^{\alpha(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for $x \in \overline{\Omega}$, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = \infty$ if $p(x) \geqslant N$;

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(f_2) $\lim_{|s|\to\infty} \frac{f(x,s)s}{|s|^{p^+}} = +\infty$ uniformly for a.e. $x \in \Omega$;

(f_3) There exists a constant $\theta \geqslant 1$, such that for any $s \in [0,1]$, $t \in \mathbb{R}$, and for each $G_\lambda \in \mathcal{F}$, and all $\mu \in [p^-, p^+]$, the inequalities $\theta G_\lambda(x,t) \geqslant G_\mu(x,st)$ hold for a.e. $x \in \Omega$.

Dirichlet problems involving the $p(x)$-Laplacian have been studied by several authors [8,10,11]. In [11, Theorems 4.7 and 4.8] the authors obtained the existence and multiplicity of solutions for (1.1) under (f_1) and

(AR) $\exists \kappa > p^+$, $R > 0$ such that $x \in \Omega$, $|s| \geqslant R \Rightarrow 0 \leqslant \kappa F(x,s) \leqslant f(x,s)s$.

That is called Ambrosetti–Rabinowitz type condition [1], in addition to the assumption as follows:

(f_4) $f(x,t) = o(|t|^{p^+-1})$, $t \to 0$ for $x \in \Omega$ uniformly;

(f_5) $f(x,-t) = -f(x,t)$, $(x,u) \in \Omega \times \mathbb{R}$.

Obviously, (f_2) can be derived from (AR). Under (AR), any (PS) sequence of the corresponding energy functional is bounded, which plays an important role of the application of variational methods. Indeed, there are many superlinear functions which do not satisfy (AR) condition. For instance when $p(x) \equiv 2, \theta \equiv 1$, the function below does not satisfy (AR)

$$f(x,t) = 2t\log\bigl(1+|t|\bigr). \tag{1.2}$$

But it is easy to see the function above (1.2) satisfies (f_1)–(f_5). While $p(x) \equiv p$ is a constant, we know the problem (1.1) with (1.2) has infinitely many solutions from [15]. In [15], the authors used the critical point theory with the Cerami condition which is weaker than the (PS) condition. In this paper the author extends the result of [15] to the $p(x)$-Laplacian equations and has obtained the existence and multiplicity of solutions for problem (1.1).

Secondly, we consider the Neumann problem based on [9]. In [16], the author has proved the infinitely many nodal solutions of the Laplace equation with Neumann boundary value problem. Here the author gives some results of the Neumann problem involving $p(x)$-Laplacian. In fact, the results of Neumann problem are similar to the Dirichlet problem in the first step.

This article is organized as follows. In Section 2, we introduce some basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega)$ and $p(x)$-Laplace operator and establish some definitions and propositions. In Section 3, we use variational techniques to prove the existence and multiplicity of solutions for the problem (1.1). In Section 4, we give some remarks on the existence of solutions for Neumann problem involving $p(x)$-Laplacian.

2. Definitions and basic properties

Let Ω be a bounded open subset of $\mathbb{R}^N$, denote $C_+(\Omega) = \{p \in C(\overline{\Omega}): \min_{x\in\overline{\Omega}} p(x) \geqslant 1\}$. For $p \in C_+(\Omega)$, denote

$$p^- = p^-(\Omega) = \min_{x\in\overline{\Omega}} p(x), \qquad p^+ = p^+(\Omega) = \max_{x\in\overline{\Omega}} p(x).$$

On the basic properties of the space $W^{1,p(x)}(\Omega)$, we refer to [11,19]. Here we display some facts which will be used later.

Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered as the same element of $\mathcal{U}(\Omega)$ when they are equal almost everywhere. For $p \in C_+(\Omega)$, define the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{u \in \mathcal{U}(\Omega): \int_\Omega \bigl|u(x)\bigr|^{p(x)}\,dx < \infty\right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0: \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

and

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega)\colon\ |\nabla u|\in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Hereafter, let

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & p(x)<N,\\ +\infty & p(x)\geqslant N.\end{cases}$$

We always assume that $p^->1$ and $p^+<p^*(x)$ for all $x\in\overline{\Omega}$.

Proposition 2.1. *(See [13,19].) If $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function and satisfies $|f(x,t)|\leqslant a(x)+b|t|^{\frac{p_1(x)}{p_2(x)}}$ for any $x\in\Omega, s\in\mathbb{R}$, where $p_1,p_2\in C_+(\overline{\Omega})$, $a\in L^{p_2(x)}(\Omega)$, $a(x)\geqslant 0$ and $b\geqslant 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $N_f(u)(x)=f(x,u(x))$ is a continuous and bounded operator.*

Proposition 2.2. *(See [13].) The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.3. *(See [13].) Set $\rho(u)=\int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$. For $u,u_k\in L^{p(x)}(\Omega)$, we have*

(1) $|u|_{p(x)}<1\ (=1;>1)\ \Leftrightarrow\ \rho(u)<1\ (=1;\ >1)$.
(2) *If* $|u|_{p(x)}>1$, *then* $|u|_{p(x)}^{p^-}\leqslant\rho(u)\leqslant|u|_{p(x)}^{p^+}$.
(3) *If* $|u|_{p(x)}<1$, *then* $|u|_{p(x)}^{p^+}\leqslant\rho(u)\leqslant|u|_{p(x)}^{p^-}$.
(4) $\lim_{k\to\infty}|u_k|_{p(x)}=0\ \Leftrightarrow\ \lim_{k\to\infty}\rho(u_k)=0$.
(5) $|u_k|_{p(x)}\to\infty\ \Leftrightarrow\ \rho(u_k)\to\infty$.

Proposition 2.4. *(See [13].) In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant c such that*

$$|u|_{L^{p(x)}(\Omega)}\leqslant c|\nabla u|_{L^{p(x)}(\Omega)},\quad \forall u\in W_0^{1,p(x)}(\Omega).$$

So $|\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. In the following discussions, we denote $\|u\|=|\nabla u|_{L^{p(x)}(\Omega)}$. Let us now introduce the $p(x)$-Laplace operator $-\Delta_{p(x)}u=-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, consider the functional

$$J(u)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x,\quad u\in X:=W_0^{1,p(x)}(\Omega),$$

$$\big(J'(u),v\big)=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v\,\mathrm{d}x,\quad \forall u,v\in X.$$

We have

Proposition 2.5. *(See [11].) The functional $J:X\to\mathbb{R}$ is convex. The mapping $J':X\to X^*$ is a strictly monotone, bounded homeomorphism, and is of type (S_+), namely, $u_n\rightharpoonup u$ in X and $\limsup_{n\to\infty}(J'(u_n)(u_n-u))\leqslant 0$ implies $u_n\to u$ in X.*

Next we give the definition of the Cerami condition which introduced by G. Cerami [3].

Definition 2.6. (See [4,5,15].) Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$, Given $c \in \mathbb{R}$, we say that Φ satisfies the Cerami c condition (we denote condition (C_c)), if

(i) any bounded sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a convergent subsequence;
(ii) there exist constants $\delta, R, \beta > 0$ such that

$$\left\|\Phi'(u)\right\| \|u\| \geqslant \beta \quad \forall u \in \Phi^{-1}\big([c-\delta, c+\delta]\big) \text{ with } \|u\| \geqslant R. \tag{2.1}$$

If $\Phi \in C^1(X, \mathbb{R})$ satisfies condition (C_c) for every $c \in \mathbb{R}$, We say that Φ satisfies condition (C).

Condition (C) is weaker than the (PS) condition. However, it was shown in [2,4] that from condition (C) it can obtain a deformation lemma, which is fundamental in order to get some minimax theorems. Thus we have

Proposition 2.7. *(See [18].) Let X a Banach space, $\phi \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$, be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \phi(u) > \phi(0) \geqslant \phi(e). \tag{2.2}$$

If ϕ satisfies the condition (C), with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \phi\big(\gamma(t)\big),$$

$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}$.
Then c is a critical value of ϕ.

We also introduce the Fountain theorem with the condition (C) which is a variant of [11,17,21]. From [20], let X be a reflexive and separable Banach space, then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_j \mid j = 1, 2, \ldots\}}, \qquad X^* = \overline{\operatorname{span}\big\{e_j^* \mid j = 1, 2, \ldots\big\}}^{w^*},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Now we write

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{2.3}$$

Proposition 2.8. *(See [15].) Assume*

(F_1) *X is a Banach space $\phi \in C^1(X, \mathbb{R})$ is an even functional, the subspaces X_k, Y_k and Z_k are defined by* (2.3)*;*
If for each $k = 1, 2, \ldots,$ there exist $\rho_k > r_k > 0$ such that
(F_2) $b_k := \inf_{u \in Z_k, \|u\|=r_k} I(u) \to +\infty$ *as* $k \to \infty$;
(F_3) $a_k := \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leqslant 0$;
(F_4) *I satisfies the condition (C).*

Then I has a sequence of critical values tending to $+\infty$.

3. Main results

In this section, we give our main results and their proofs. Firstly, we give the definition of the weak solution for (1.1).

Definition 3.1. $u \in X$ is called a weak solution of (1.1), if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, \mathrm{d}x = \int_\Omega f(x,u)v \, \mathrm{d}x, \quad \forall v \in X = W_0^{1,p(x)}(\Omega).$$

Let

$$I(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \, \mathrm{d}x - \int_\Omega F(x,u) \, \mathrm{d}x,$$

where

$$F(x,t) = \int_0^t f(x,s) \, \mathrm{d}s \quad \text{for } x \in \Omega, \text{ and } t \in \mathbb{R},$$

then it is easy to see that $I \in C^1(X,\mathbb{R})$ and

$$I'(u)v = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, \mathrm{d}x - \int_\Omega f(x,u)v \, \mathrm{d}x, \quad \forall u, v \in X$$

(see [11]). We write

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \, \mathrm{d}x, \qquad \Psi(u) = \int_\Omega F(x,u) \, \mathrm{d}x,$$

then $I = J - \Psi$.

We give a lemma which plays the most important role.

Lemma 3.2. *Under the assumptions* (f_1)–(f_3), $I(u)$ *satisfies the condition* (C).

Proof. For all $c \in \mathbb{R}$, we show that I satisfies (i) of condition (C). Let $\{u_n\} \subset X$ be bounded, $I(u_n) \to c$ and $I'(u_n) \to 0$. Hence $\{u_n\}$ has a weakly convergent subsequence in X. Without loss of generality, we assume that $u_n \rightharpoonup u$. $\Psi(u) = \int_\Omega F(x,u) \, \mathrm{d}x$, then $\Psi' : X \to X^*$ is completely continuous because of assumption (f_1) from [11]. Hence $\Psi'(u_n) \to \Psi'(u)$. Since $I'(u_n) = J'(u_n) - \Psi'(u_n) \to 0$, $J'(u_n) \to \Psi'(u)$, and J' is a homeomorphism in view of Proposition 2.5, we know $u_n \to u$ in X.

Now check that I satisfies (ii) of condition (C) too. We argue by contradiction. Were the statement false, there exist $c \in \mathbb{R}$ and $\{u_n\} \subset X$ satisfying:

$$I(u_n) \to c, \qquad \|u_n\| \to +\infty, \qquad \big\|I'(u_n)\big\| \|u_n\| \to 0. \tag{3.1}$$

We can choose $\|u_n\| > 1$, for $n \in \mathbb{N}$, thus

$$c = \lim_{n\to\infty}\left\{I(u_n) - \frac{1}{\overline{p}_n}\big\langle I'(u_n), u_n\big\rangle\right\} = \lim_{n\to\infty} \frac{1}{\overline{p}_n}\int_\Omega f(x,u_n)u_n \, \mathrm{d}x - \int_\Omega F(x,u_n) \, \mathrm{d}x, \tag{3.2}$$

where

$$\overline{p}_n = \frac{\int_\Omega |\nabla u_n|^{p(x)} \, \mathrm{d}x}{\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)} \, \mathrm{d}x}.$$

Denote $\omega_n = \frac{u_n}{\|u_n\|}$, then $\|\omega_n\| = 1$. Up to subsequences, for some $\omega \in X$, we get

$$\begin{aligned} &\omega_n \rightharpoonup \omega && \text{in } X, \\ &\omega_n \to \omega && \text{in } L^{p^+}, \\ &\omega_n(x) \to \omega(x) && \text{a.e. in } \Omega. \end{aligned} \tag{3.3}$$

If $\omega(x) \equiv 0$, as in [14], we can define a sequence $\{t_n\} \subset \mathbb{R}$:

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n). \tag{3.4}$$

If there is a number of t_n satisfying (3.4), one choose one of them. Fix any $m > 0$, let $\bar{\omega}_n = (2p^+m)^{\frac{1}{p^-}}\omega_n$, since $\omega_n \rightharpoonup \omega \equiv 0$, and $\Psi(u)$ is weakly continuous from [11],

$$\lim_{n\to\infty} \int_\Omega F(x, \bar{\omega}_n)\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega F\big(x, \big(2p^+m\big)^{\frac{1}{p^-}}\omega_n\big)\,\mathrm{d}x = 0. \tag{3.5}$$

Then for n large enough

$$\begin{aligned} I(t_n u_n) &\geqslant I(\bar{\omega}_n) \\ &= \int_\Omega \frac{1}{p(x)}\big|\nabla\big(2p^+m\big)^{\frac{1}{p^-}}\omega_n\big|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x, \bar{\omega}_n)\,\mathrm{d}x \\ &\geqslant \int_\Omega \frac{1}{p^+}\big(2p^+m\big)|\nabla\omega_n|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x, \bar{\omega}_n)\,\mathrm{d}x \\ &= 2m - \int_\Omega F(x, \bar{\omega}_n)\,\mathrm{d}x \geqslant m, \end{aligned} \tag{3.6}$$

that is, $\lim_{n\to\infty} I(t_n u_n) = +\infty$. Since $I(0) = 0$, and $I(u_n) \to c$, then $0 < t_n < 1$, n large enough. We have

$$\int_\Omega \big(\big|\nabla(t_n u_n)\big|^{p(x)}\,\mathrm{d}x - f(x, t_n u_n)t_n u_n\big)\,\mathrm{d}x = \big\langle I'(t_n u_n), t_n u_n\big\rangle = t_n \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_n} I(tu_n) = 0. \tag{3.7}$$

We obtain that

$$\begin{aligned} \int_\Omega \bigg(\frac{1}{\overline{p}_{t_n}} f(x, t_n u_n)t_n u_n - F(x, t_n u_n)\bigg)\,\mathrm{d}x &= \int_\Omega \bigg(\frac{1}{\overline{p}_{t_n}}|\nabla t_n u_n|^{p(x)} - F(x, t_n u_n)\bigg)\,\mathrm{d}x \\ &= I(t_n u_n) \to +\infty \quad \text{as } n \to \infty \text{ by (3.6)}, \end{aligned} \tag{3.8}$$

where

$$\overline{p}_{t_n} = \frac{\int_\Omega |\nabla(t_n u_n)|^{p(x)}\,\mathrm{d}x}{\int_\Omega \frac{1}{p(x)}|\nabla(t_n u_n)|^{p(x)}\,\mathrm{d}x}.$$

Let $\lambda_{u_n} = \overline{p}_n$, $\lambda_{t_n u_n} = \overline{p}_{t_n}$, then $\lambda_{u_n}, \lambda_{t_n u_n} \in [p^-, p^+]$, hence, $G_{\lambda_{u_n}}, G_{\lambda_{t_n u_n}} \in \mathcal{F}$.

$$\begin{aligned} \int_\Omega \bigg(\frac{1}{\overline{p}_n} f(x, u_n)u_n - F(x, u_n)\bigg)\,\mathrm{d}x &= \frac{1}{\overline{p}_n}\int_\Omega G_{\lambda_{u_n}}(x, u_n)\,\mathrm{d}x \geqslant \frac{1}{\overline{p}_n\theta}\int_\Omega G_{\lambda_{t_n u_n}}(x, t_n u_n)\,\mathrm{d}x \quad \text{by } (f_3) \\ &= \frac{\overline{p}_{t_n}}{\overline{p}_n\theta}\int_\Omega \bigg(\frac{1}{\overline{p}_{t_n}} f(x, t_n u_n) - F(x, t_n u_n)\bigg)\,\mathrm{d}x \to +\infty, \end{aligned} \tag{3.9}$$

since $\inf_n \frac{\overline{p}_{t_n}}{\overline{p}_n\theta} > 0$, which contradicts (3.2).

If $\omega(x) \not\equiv 0$, by(3.1), we can get

$$\int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega f(x, u_n)u_n\,\mathrm{d}x = \big\langle I'(u_n), u_n\big\rangle = o(1)\|u_n\|, \tag{3.10}$$

that is,

$$1 - o(1) = \int_\Omega \frac{f(x, u_n)u_n}{\rho(u_n)}\,\mathrm{d}x \geqslant \int_\Omega \frac{f(x, u_n)u_n}{\|u_n\|^{p^+}}\,\mathrm{d}x = \int_\Omega \frac{f(x, u_n)u_n}{|u_n|^{p^+}}|\omega_n|^{p^+}\,\mathrm{d}x, \tag{3.11}$$

where $\rho(u) = \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x$, and by Proposition 2.3.

Let $\Omega_0 := \{x \in \Omega:\ \omega(x) = 0\}$, then for $x \in \Omega \setminus \Omega_0 = \{x \in \Omega:\ \omega(x) \neq 0\}$, we have $|u_n(x)| \to +\infty$ as $n \to \infty$, by (f_2)

$$\frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\omega_n|^{p^+} \to +\infty \quad \text{as } n \to \infty.$$

Using Fatou lemma, in view of $|\Omega \setminus \Omega_0| > 0$, we have

$$\int\limits_{\Omega\setminus\Omega_0} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\omega_n|^{p^+}\,\mathrm{d}x \to +\infty \quad \text{as } n \to \infty. \tag{3.12}$$

On the other hand, by (f_1) and (f_2), there exists $\gamma > -\infty$, such that $\frac{f(x,s)s}{s^{p^+}} \geqslant \gamma$ for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Moreover, we have

$$\int\limits_{\Omega_0} |\omega_n|^{p^+}\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Now there exists $\Lambda > -\infty$ such that

$$\int\limits_{\Omega_0} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\omega_n|^{p^+}\,\mathrm{d}x \geqslant \gamma \int\limits_{\Omega_0} |\omega_n|^{p^+}\,\mathrm{d}x \geqslant \Lambda > -\infty. \tag{3.14}$$

Combining (3.11), (3.12) and (3.14), there is a contradiction, which proves that I satisfies condition (C). □

Remark 3.3. In the proof of Lemma 3.2, (f_3) plays a crucial role. It is easy to prove that if $f(x,t)$ is increasing in t, then (AR) implies (f_3) when t is enough large. Indeed, we can take $\theta = \frac{1}{1-\frac{p^+}{\kappa}} > 1$, then $\theta G_\lambda(x,t) - G_\mu(x,st) \geqslant f(x,t)t - f(x,st)st \geqslant 0$. But generally speaking, (AR) does not implies (f_3), see following Example 3.4.

Example 3.4. Let $p(x) \equiv p$. Set

$$f(x,t) = (p+2)|t|^p t + (p+1)|t|^{p-1} t \sin^2\frac{1}{t} - |t|^{p-1}\sin\frac{1}{t}\cos\frac{1}{t}.$$

By a simple calculation, we get $F(x,t) = |t|^{p+2} + |t|^{p+1}\sin^2\frac{1}{t} > 0$, if $t \neq 0$. Let $\kappa = p+1 > p$, then there exists $R > 0$, such that

$$|t| > R, \quad x \in \Omega \quad \Rightarrow \quad f(x,t)t - \kappa F(x,t) = |t|^{p+2} - 2|t|^{p-1} t \sin\frac{1}{t}\cos\frac{1}{t} > 0.$$

That is, $f(x,t)$ satisfies (AR) condition. But since $p(x) \equiv p$, $\mathcal{F}$ contains only one element. Hence we have

$$G(x,t) = f(x,t)t - pF(x,t) = 2|t|^{p+2} + |t|^{p+1}\sin^2\frac{1}{t} - 2|t|^{p-1} t \sin\frac{1}{t}\cos\frac{1}{t} \in \mathcal{F}.$$

But $\forall \theta \geqslant 1$, there exist $t_0 = \frac{1}{n_0\pi}$ and $s_0 = \frac{n_0}{n_0+\frac{1}{2}} \in (\frac{1}{2}, 1) \subset [0,1]$, where $n_0 \in \mathbb{Z}_+$ and $n_0 > (2^{p+2}\theta - 1)$. Then we have

$$\begin{aligned}
\theta G(x,t_0) - G(x,s_0t_0) &= 2\theta t_0^{p+2} - 2s_0^{p+2}t_0^{p+2} - s_0^{p+1}t_0^{p+1} \\
&\leqslant 2\left(\theta - \frac{1}{2^{p+2}}\right)t_0^{p+2} - \frac{1}{2^{p+1}}t_0^{p+1} \\
&= \left[\left(2^{p+2}\theta - 1\right)t_0 - 1\right]\left(\frac{t_0}{2}\right)^{p+1} \\
&< \left(\frac{2^{p+2}\theta - 1}{2^{p+2}\theta - 1} - 1\right) = 0,
\end{aligned}$$

for all $x \in \Omega$ hold. Moreover, f does not satisfy (f_3).

Our main results are the following two theorems:

Theorem 3.5. *If f satisfies (f_1)–(f_4), then the problem* (1.1) *has at least one nontrivial solution.*

Proof. We only show that $I(u)$ satisfies conditions of Proposition 2.7. By Lemma 3.2, I satisfies conditions (C) in X, from (f_2) $p^+ < \alpha^- \leqslant \alpha(x) < p^*$. From [11], there are $r, \sigma > 0$ such that $I(u) \geqslant \sigma > 0$, for every $u \in X$ and $\|u\| = r$. As the proof in [18], we can find $u_0 \in X \setminus \overline{B}(0, r)$ such that $I(u_0) < 0$. Thus I has at least one nontrivial critical value, i.e. the problem (1.1) has a nontrivial solution. □

Remark 3.6. When $f(x,t) \geqslant 0, t \geqslant 0$, the problem (1.1)has a positive solution u, i.e. $u(x) > 0$ for $x \in \Omega$, unless $u(x) \equiv 0$ on Ω by Theorem 3.5 and Proposition 3.4 in [8].

Theorem 3.7. *If f satisfies (f_1)–(f_3) and (f_5), then the problem* (1.1) *has infinite many pairs of solutions.*

Proof. According to Lemma 3.2 and (f_5), I is an even functional and satisfies condition (C). It is sufficient to prove that for k large enough there exist $\rho_k > r_k > 0$ such that

(B_1) $b_k := \inf\{I(u) \mid u \in Z_k,\ \|u\| = r_k\} \to +\infty$;
(B_2) $a_k := \max\{I(u) \mid u \in Y_k,\ \|u\| = \rho_k\} \leqslant 0$ $(k \to \infty)$.

(B_1) was obtained from [11]. Now we only show (B_2) is valid. Since $\dim Y_k < +\infty$, and when $\|u\| \geqslant 1$

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, \mathrm{d}x \leqslant \frac{1}{p^-} \int_\Omega |\nabla u|^{p(x)}\, \mathrm{d}x \leqslant \frac{1}{p^-} \|u\|^{p^+} \leqslant C_k |u|_{p^+}^{p^+}.$$

From (f_2), there exist $R_k > 0$, $|s| \geqslant R_k$, $F(x,s) \geqslant 2C_k |s|^{p^+}$. Using the same method in [15], we easily find that $I(u) \to -\infty$, as $\|u\| \to +\infty$ for $u \in Y_k$. This completes the proof. □

4. Remarks on Neumann problem

In this section, we give some remarks on the existence and multiplicity of solutions for the $p(x)$-Laplacian Neumann problem of the following type:

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2} u = f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω as above, n is the outward normal vector of $\partial\Omega$. The weak solution of problem (4.1) is a critical point of the following functional:

$$\begin{aligned} I(u) &= \int_\Omega \frac{1}{p(x)} \big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\, \mathrm{d}x - \int_\Omega F(x,u)\, \mathrm{d}x \\ &=: J(u) - \Psi(u), \quad u \in Y := W^{1,p(x)}(\Omega). \end{aligned} \tag{4.2}$$

From [9] we know that $I \in C^1(Y, \mathbb{R})$ and $I' : Y \to Y^*$, then

$$\begin{aligned} \big\langle I'(u), v \big\rangle &= \int_\Omega \big(|\nabla u|^{p(x)-2} \nabla u \nabla v\, \mathrm{d}x + |\nabla u|^{p(x)-2} u v\big)\, \mathrm{d}x - \int_\Omega f(x,u) v\, \mathrm{d}x \\ &=: \big\langle J'(u), v \big\rangle - \big\langle \Psi'(u), v \big\rangle, \quad v \in Y. \end{aligned} \tag{4.3}$$

The following proposition exposes the properties of $J' : Y \to Y^*$.

Proposition 4.1. *(See [9].)*

(i) *$J' : Y \to Y^*$ is a continuous, bounded and strictly monotone operator*;
(ii) *J' is a mapping of type (S_+), moreover J' is a homeomorphism.*

From Proposition 4.1, we have

Lemma 4.2. *If f satisfies (f_1)–(f_3) then I satisfies condition (C).*

Theorem 4.3. *Under the assumptions (f_1)–(f_4),* (4.1) *has a nontrivial solution.*

Theorem 4.4. *If f fulfills (f_1)–(f_3) and (f_5), then* (4.1) *has infinitely many pairs of solutions.*

The proofs of Lemma 4.2, Theorem 4.3 and Theorem 4.4 are similar to those of Lemma 3.2, Theorem 3.5 and Theorem 3.7, respectively, so we omit them here.

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